Journal of Statistical Physics, Vol. 46, Nos. 1/2, 1987

More Correlation Inequalities for a Class of Even Ferromagnets

Kei-ichi Kondo[1,2]

Received March 28, 1986; revision received September 3, 1986

Rigorous correlation inequalities are presented for a class of even ferromagnets, which includes the spin-1/2 Ising model and scalar φ^4 models. One of them leads to an extension of the Glimm and Jaffe uniform upper bound on the φ^4 renormalized coupling constant into the nonsymmetric regime.

KEY WORDS: Correlation inequality; Ising model; scalar field model.

1. INTRODUCTION

In previous work,[1] a series of new correlation inequalities were presented for ferromagnets with the pair interaction Hamiltonian and single spin measure belonging to the Ellis–Monroe–Newman class.[2]

In this paper, I report more correlation inequalities for the fourth Ursell function (or cumulant) U_4 with the presence of the external magnetic field $h \geqslant 0$. Note that the previous work omitted terms containing at least one expectation $\langle \varphi(x_1) \cdots \varphi(x_n) \rangle$ with n odd, because we aimed at obtaining bounds on the four- (and six-) point coupling constants in the single-phase region.

Correlation inequalities are used extensively in the triviality proof[3,4] of $(\varphi^4)\, d$ theories in $d > 4$ dimensions constructed as subsequence limits of the corresponding lattice models. The triviality implies that the renormalized coupling constant defined by $g^{(4)} \equiv -\bar{U}_4/(\chi^2 \xi^d)$ vanishes as the lattice system is moved to the critical point J_c, which was only proven in the

[1] Department of Physics, Faculty of Science, Nagoya University, Chikusa-ku, Nagoya 464, Japan.

[2] Present address: Department of Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan.

0022-4715/87/0100-0035$05.00/0

case of approaching the critical point J_c from the high-temperature phase $J<J_c$, $h=0$. The absolute bound[4,5] on $g^{(4)}$ is also known only in the single-phase region.

The inequality (2.2) given in the next section is sufficient to prove the absolute *upper* bound on $g^{(4)}$, i.e., for all $J \geqslant 0$ and all $h \geqslant 0$,

$$g^{(4)} \equiv -\bar{U}_4/(\chi^2 \xi^d) \leqslant \text{const}(d) \tag{1.1}$$

where the constant is independent of J, h, and all parameters used to specify the single spin measure $\nu(d\varphi)$ (if it belongs to the Ellis–Monroe–Newman class). It should be remarked that, because of the possible violation of the Lebowitz inequality $U_4 \leqslant 0$ [see (2.1)], we have no simple lower bound such that $g^{(4)} \geqslant 0$, which will be satisfied only at the critical point J_c.

However, the absolute upper bound on $g^{(4)}$ in the whole (J, h) plane may support the idea of critical point dominance,[6,7] following Glimm and Jaffe.[6]

We also obtained correlation inequalities for U_6. But their form is so complicated that they are not reported in this paper.

Explicit forms of the cumulants are given as follows:

$$U_2(x_1, x_2) \equiv G_2(x_1, x_2) = \langle x_1, x_2 \rangle - \langle x_1 \rangle \langle x_2 \rangle \tag{1.2}$$

$$U_3(x_1, x_2, x_3) = \langle x_1, x_2, x_3 \rangle - \sum \langle x_{i_1} \rangle \langle x_{i_2}, x_{i_3} \rangle + 2\langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \tag{1.3}$$

$$\begin{aligned} U_4(x_1,..., x_4) = {} & \langle x_1, x_2, x_3, x_4 \rangle - \sum \langle x_{i_1}, x_{i_2} \rangle \langle x_{i_3}, x_{i_4} \rangle \\ & - \sum \langle x_{i_1} \rangle \langle x_{i_2}, x_{i_3}, x_{i_4} \rangle + 2 \sum \langle x_{i_1} \rangle \langle x_{i_2} \rangle \langle x_{i_3}, x_{i_4} \rangle \\ & - 6\langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \langle x_4 \rangle \end{aligned} \tag{1.4}$$

where we used the simplified notation

$$\langle x_1,..., x_n \rangle \equiv \langle \varphi(x_1) \cdots \varphi(x_n) \rangle \tag{1.5}$$

Note that the absolute bound on the three-point amplitude was already obtained in Ref. 8, which can be shown for our models using the correlation inequalities (3.12a) and (3.12b) in Ref. 1:

$$0 \geqslant U_3(i, j, k) \geqslant -4\langle i \rangle \, G_2(j, k) \tag{1.6}$$

Let $\Phi = \{\varphi_i \in \mathbb{R};\ i=1,..., N\}$ be a finite family of real-valued, random variables, whose joint distribution μ on $\mathbb{R}^N$ is given by

$$d\mu_{J,h}(\Phi) = Z_{J,h}^{-1} \exp[-H_{J,h}(\Phi)] \prod_{i=1}^{N} d\nu(\varphi_i)$$

where $H_{J,h}(\Phi)$, $(J, h) = \{J_{ij}, h_i\}$, is the Hamiltonian defined by

$$H_{J,h}(\Phi) = -\sum_{1 \leqslant i \leqslant j \leqslant N} J_{ij}\varphi_i\varphi_j - \sum_{1 \leqslant i \leqslant N} h_i\varphi_i, \qquad J_{ij} \geqslant 0, \quad h_i \geqslant 0$$

and $Z_{J,h}$ is the partition function, chosen so that $\int d\mu_{J,h}(\Phi) = 1$.

Consider the fourfold duplicate system whose random variables $\Phi^{(a)}$ $(a = 1,..., 4)$ are independently, identically distributed by μ. If the single spin measure ν belongs to the Ellis–Monroe–Newman class, then, for arbitrary sets of four multi-indices $P = (P(1),..., P(4))$,

$$\int d\mu(\Phi^{(1)}) \cdots d\mu(\Phi^{(4)}) \prod_{a=1}^{4} [(B\Phi)^{(a)}]^{P(a)} \geqslant 0 \tag{1.7}$$

where $(B\Phi)^{(a)} = \sum_{b=1}^{4} B_{ab}\Phi^{(b)}$ and B is the orthogonal matrix

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Note that each $p_k(a)$ $(a = 1,..., 4;\ k = 1,..., N)$ takes nonnegative integral values. For details see Ref. 1.

2. NEW CORRELATION INEQUALITIES

Now I give new correlation inequalities, together with the corresponding sets of multi-indices $P = (P(1),..., P(4))$. In the following, $P(a) = n$ implies that $p_{k_1}(a) = 1$, $p_{k_2}(a) = 1,..., p_{k_n}(a) = 1$ for arbitrarily chosen sites $k_1,..., k_n$.

1. $P = (1, 1, 1, 1)$:

$$U_4(i, j, k, l) \leqslant -4\langle i\rangle\, U_3(j, k, l) \tag{2.1}$$

2. $P = (0, 2, 2, 0)$, $(0, 2, 0, 2)$, $(0, 0, 2, 2)$:

$$U_4(i, j, k, l) \geqslant -4G_2(i, j)\, G_2(k, l) \tag{2.2}$$

3. $P = (2, 2, 0, 0,)$, $(2, 0, 2, 0)$, $(2, 0, 0, 2)$:

$$U_4(i, j, k, l) \geqslant -4\langle i\rangle\, U_3(j, k, l) - 4\langle j\rangle\, U_3(i, k, l) - 4G_2(i, j)\, G_2(k, l) - 16\langle i\rangle\langle j\rangle\, G_2(k, l) \tag{2.3}$$

Remark 1. For $P=(0, 4, 0, 0)$, $(0, 0, 4, 0)$, and $(0, 0, 0, 4)$, we get

$$U_4(i, j, k, l) \geqslant -4G_2(i, j)\, G_2(k, l) - 4G_2(i, k)\, G_2(j, l) - 4G_2(i, l)\, G_2(j, k)$$

But this is weaker than (2.2), in view of Griffiths second inequality ($G_2 \geqslant 0$).

Remark 2.[3] The inequality (1.6) [(3.12b) of Ref. 1] with factor 2 instead of 4 can be obtained from the Ginibre inequality[9] $\langle t_i q_j q_k \rangle \geqslant 0$. So the EMN argument is weaker here. This is the case for the inequality (3.13b) of Ref. 1.

ACKNOWLEDGMENTS

I would like to thank H. Nakatani for his help in computer programming and Alan Sokal for his interest in this work. The author is a Fellow of the Japan Society for the Promotion of Science. This work was supported in part by a Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture (60790026).

REFERENCES

1. K.-I. Kondo, T. Otofuji, and Y. Sugiyama, *J. Stat. Phys.* **40**:563 (1985).
2. R. S. Ellis and J. L. Monroe, *Commun. Math. Phys.* **41**:33 (1975); R. S. Ellis, J. L. Monroe, and C. M. Newman, *Commun. Math. Phys.* **46**:167 (1976); R. S. Ellis and C. M. Newman, *Trans. Am. Math. Soc.* **237**:83 (1978).
3. J. Fröhlich, *Nucl. Phys. B* **200**(FS4):281 (1982).
4. M. Aizenman, *Phys. Rev. Lett.* **47**:1 (1981); M. Aizenman, *Commun. Math. Phys.* **86**:1 (1982).
5. J. Glimm and A. Jaffe, *Ann. Inst. Henri Poincaré A* **22**:97 (1975).
6. J. Glimm and A. Jaffe, *Ann. Inst. Henri Poincaré* **21**:27 (1974).
7. A. S. Wightman, in *The Whys of Subnuclear Physics* (1977 Erice Lectures), A. Zichichi, ed. (Plenum Press, New York, 1979), pp. 983–1025; A. S. Wightman, in *New Developments in Quantum Field Theory and Statistical Mechanics*, M. Levy and P. K. Mitter, eds. (Plenum Press, New York, 1977).
8. A. D. Sokal, *J. Stat. Phys.* **25**:25 (1981).
9. G. S. Sylvester, *J. Stat. Phys.* **15**:327 (1976); J. Ginibre, *Commun. Math. Phys.* **16**:310 (1976).

[3] This remark is due to Alan Sokal.